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THE CRITICAL MODES OF OSCILLATION OF A TRANSVERSELY INHOMOGENEOUS PLATE WITH A PERIODIC STRUCTURE†

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A problem on the harmonic oscillations of a transversely inhomogeneous plate, the physicomechanical properties of which are described by arbitrary piecewise-continuous functions, is considered using the three-dimensional equations of the theory of elasticity. A general representation of the solution is found and, on the basis of this solution by a separation of variables, the initial problem, in the case of homogeneous boundary conditions on the faces of the plate, is reduced to two eigenvalue problems with a pair of eigenvalue parameters. The angular frequency and wave number perform the role of these parameters. Particular attention is paid to the critical frequencies for which multiple eigenvalues exist in the wave-number spectrum. A classification of them is given and differential equations are obtained which describe the distribution of the critical modes in the domain occupied by the plate. The general theory constructed, together with the Floquet–Lyapunov theory, is applied to plates with a periodic transverse inhomogeneity. Calculations are carried out for a finely stratified plate with alternating rigid (steel) and soft (rubber) layers.

The investigation of the propagation of harmonic waves in semi-bounded bodies and the oscillation of plates reduces to eigenvalue problems in the pair of parameters (k, ω), where k is the wave number and ω is the angular frequency. In the majority of papers concerned with this problem, as a rule the question of the mathematical description of the set of modes corresponding to a multiple wave number was left aside. Various terms are used for the value of the frequency in such cases: the critical frequency, the stop frequency, the cut-off frequency and the resonance frequency. Using the example of an anisotropic, transversely inhomogeneous strip, it was shown in [1, 2] that critical modes can have a power-low growth, and this result was extended in [3–6] to arbitrary solid waveguides with a single axis of propagation of normal modes. A full classification of the critical frequencies of a homogeneous plate has been given in [7, 8] in relation to the investigation of the interaction of the dispersion and dissipative decay of normal modes and two-dimensional equations have been obtained for describing the distribution of the critical modes. A similar result has been announced for the case of a transversely inhomogeneous plate made of elastic [9] and piezo-active materials [10]. The results of an investigation of the problem for a finely stratified plate having a periodic structure are described below.

1. GENERAL REPRESENTATION OF THE SOLUTION

We will denote the domain occupied by the plate by $V = S \times [z^-, z^+]$, where $x_1, x_2 \in S$, $x_3 = z \in \infty$ $[z^-, z^+]$. We consider the equations for the steady oscillations of a one-dimensionally inhomogeneous, transversally isotropic medium in the domain V. We have

$$\begin{gathered}
\partial(c_{44}\partial u_1 + c_{44}\partial_1 u_3) + c_{13}\partial\partial_1 u_3 + (c_{11}\partial_1^2 + c_{66}\partial_2^2)u_1 + (c_{12} + c_{66})\partial_1\partial_2 u_2 + \rho\omega^2 u_1 = 0 \\
\partial(c_{44}\partial u_2 + c_{44}\partial_2 u_3) + c_{13}\partial\partial_2 u_3 + (c_{12} + c_{66})\partial_1\partial_2 u_1 + (c_{66}\partial_1^2 + c_{11}\partial_2^2)u_2 + \rho\omega^2 u_2 = 0 \\
\partial(c_{33}\partial u_3 + c_{13}\partial_1 u_1 + c_{13}\partial_2 u_2) + c_{44}\partial\partial_1 u_1 + c_{44}\partial\partial_2 u_2 + c_{44}\Delta u_3 + \rho\omega^2 u_3 = 0 \qquad (1.1) \\
\partial_\alpha = \partial/\partial x_\alpha, \quad \partial = \partial/\partial z, \quad \Delta = \partial_1^2 + \partial_2^2 \ \ (\alpha = 1, 2), \quad 2c_{66} = c_{11} - c_{12}
\end{gathered}$$

Here u_j are the amplitudes of the displacements ($j = 1, 2, 3$), $c_{ij} = c_{ij}(z)$ and $\rho = \rho(z)$ are the moduli of elasticity and the density, respectively, regarding which it is assumed that they are piecewise-continuous functions of z, and ω is the angular frequency of the harmonic oscillations. In the case of an isotropic material

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$$c_{11} = c_{33} = \lambda + 2\mu, \qquad c_{44} = c_{66} = \mu, \qquad c_{12} = c_{13} = \lambda$$

where λ and μ are the Lamé elastic parameters.

We will represent the plane field $\mathbf{u}^0 = \{u_1, u_2\}$ in the form of potential and vortex parts, putting

$$u_1 = \partial_1 v_1 + \partial_2 v_2, \qquad u_2 = \partial_2 v_1 - \partial_1 v_2 \tag{1.2}$$

Substituting Eqs (1.2) into system (1.1) and transforming the first two equations, we obtain

$$\begin{gathered} \partial_1 L_1(v_1, u_3) + \partial_2 L(v_2) = 0, \qquad \partial_2 L_1(v_1, u_3) - \partial_1 L(v_2) = 0 \\ L_1(v_1, u_3) = \partial(c_{44}\partial v_1 + c_{44}u_3) + c_{13}\partial u_3 + c_{11}\Delta v_1 + \rho\omega^2 v_1 \\ L(v_2) = \partial(c_{44}\partial v_2) + c_{66}\Delta v_2 + \rho\omega^2 v_2 \end{gathered} \tag{1.3}$$

and the third equation of (1.1) gives

$$L_2(v_1, u_3) = \partial(c_{33}\Delta u_3 + c_{13}\Delta v_1) + c_{44}(\partial\Delta v_1 + \Delta u_3) + \rho\omega^2 u_3 = 0$$

Since relations (1.3) are identical to the Cauchy–Riemann identities for adjoint harmonic functions, the initial system of equations (1.1) is equivalent to the following

$$L(v_2) = g_2, \qquad L_1(v_1, u_3) = g_1, \qquad L_2(v_1, u_3) = 0 \tag{1.4}$$

where g_1 and g_2 are adjoint harmonic functions of the variables x_1 and x_2 which, generally speaking, also depend on z. The following can be taken as a particular solution of the inhomogeneous system of equations (1.4)

$$v_2 = v_2^0, \qquad v_1 = v_1^0, \qquad u_3^0 = 0 \tag{1.5}$$

where v_1^2 and v_2^0 are solutions of the equations

$$\partial(c_{44}\partial v_\alpha^0) + \rho\omega^2 v_\alpha^0 = g_\alpha$$

in the case of arbitrary boundary conditions on the faces $z = z^\pm$. Using this arbitrariness, it is always possible, for any value of ω, to make problems of constructing v_α^0 invertible and to represent their solutions in the form

$$v_\alpha^0 = \int_{z^-}^{z^+} K(z - y) g_\alpha(x_1, x_2, y)\, dy \tag{1.6}$$

The equality

$$u_1^0 = \partial_1 v_1^0 + \partial_2 v_2^0 = 0, \qquad u_2^0 = \partial_2 v_1^0 - \partial_1 v_2^0 = 0 \tag{1.7}$$

follows from expressions (1.2), (1.3) and (1.6), and, consequently, the displacement fields corresponding to the particular solution are identically equal to zero. One can therefore put $g_1 = g_2 = 0$. Hence, the initial system of equations (1.1) is separated into an equation in the function v_2

$$L(v_2) = 0 \tag{1.8}$$

and a system of equations in the pair of functions v_1, u_3

$$L_\alpha(v_1, u_3,) = 0, \qquad \alpha = 1, 2 \tag{1.9}$$

We shall assume that, when $z = z^\pm$, the following boundary conditions are specified

$$\begin{gathered} \sigma_{\alpha 3}\big|_{z=z^\pm} = (c_{44}\partial u_1 + c_{44}\partial_1 u_3)\big|_{z=z^\pm} = q_\alpha^\pm(x_1, x_2) \\ \sigma_{33}\big|_{z=z^\pm} = (c_{33}\partial u_3 + c_{13}\partial_1 u_1 + c_{13}\partial_2 u_2)\big|_{z=z^\pm} = q^\pm(x_1, x_2) \end{gathered} \tag{1.10}$$

Here, σ_{ij} are the components of the stress tensor.

We shall show that the boundary conditions presented above can also be separated in the case of an inhomogeneous layer to obtain disconnected boundary-value problems in the function v_2 and the functions v_1 and u_3.

In fact, we will represent the functions q_1 and q_2 in the form

$$q_1^{\pm} = \partial_1\tau_1^{\pm} + \partial_2\tau_2^{\pm}, \qquad q_2^{\pm} = \partial_2\tau_1^{\pm} - \partial_1\tau_2^{\pm} \tag{1.11}$$

Substituting expressions (1.2) and (1.11) into conditions (1.10), we obtain after some reduction

$$\begin{aligned} &M(v_2)|_{z=z^{\pm}} \equiv c_{44}\partial v_2|_{z=z^{\pm}} = \tau_2^{\pm} + f_2^{\pm} \\ &M_1(v_1, u_3)|_{z=z^{\pm}} \equiv [c_{44}(\partial v_1 + u_3)]|_{z=z^{\pm}} = \tau_1^{\pm} + f_1^{\pm} \\ &M_2(v_1, u_3)|_{z=z^{\pm}} \equiv (c_{33}\partial u_3 + c_{13}\Delta v_1)|_{z=z^{\pm}} = q^{\pm} \end{aligned} \tag{1.12}$$

Here $f_1^{\pm}(x_1, x_2)$ and $f_2^{\pm}(x_1, x_2)$ are adjoint harmonic functions.

It can be shown that, for all values of ω for which the problem

$$\partial[c_{44}\partial\phi(z)] + \rho\omega^2\phi(z) = 0, \qquad \partial\phi(\pm h) = 0 \tag{1.13}$$

has only a trivial solution, it is possible to put $f_{\alpha}^{\pm}(x_1, x_2) = 0$. Values of ω for which problem (1.13) has a non-trivial solution belong to the category of critical values, and the corresponding solutions will be constructed below.

2. HOMOGENEOUS SOLUTIONS

We equate the right-hand sides in boundary conditions (1.12) to zero and we shall seek solutions of the corresponding homogeneous boundary-value problems in the form

$$\begin{aligned} &v_2 = a(z)m_2(x_1, x_2), \quad v_1 = a_1(z)m_1(x_1, x_2), \quad u_3 = ika_2(z)m_1(x_1, x_2) \\ &\Delta m_{\alpha} + k^2 m_{\alpha} = 0 \end{aligned} \tag{2.1}$$

Substituting expressions (2.1) into equations (1.8) and (1.9) and the boundary conditions (1.12) and separating the variables, we obtain the eigenvalue problems

$$(c_{44}\mathbf{a}')' + (\rho\omega^2 - k^2c_{44})\mathbf{a} = 0, \qquad \mathbf{a}'(z^{\pm}) = 0 \tag{2.2}$$

$$(\mathbf{Ca}')' + ik[(\mathbf{Ba})' + \mathbf{B}^*\mathbf{a}'] - k^2\mathbf{Aa} + \rho\omega^2\mathbf{a} = 0 \quad (\mathbf{Ca}' + ik\mathbf{Ba})|_{z=z^{\pm}} = 0 \tag{2.3}$$

Here $\mathbf{a} = \{a_1, a_2\}$ is a vector function, a prime denotes differentiation with respect to z and $\mathbf{C} = ||C_{ij}||$, $\mathbf{B} = ||B_{ij}||$ and $\mathbf{A} = ||A_{ij}||$ are matrix functions ($\mathbf{B}^*$ is the transpose of the matrix $\mathbf{B}$) with the following non-zero elements:

$$C_{22} = c_{33}, \quad B_{12} = c_{44}, \quad B_{21} = c_{13}, \quad A_{11} = c_{11}, \quad A_{22} = c_{44}$$

We will denote the eigenvalues of problems (2.2) and (2.3) with respect to the parameter k by Λ_1 and Λ_2.

Assertion 1. For any real value of the frequency ω, the spectrum $\Lambda = \Lambda_{\beta r} \cup \Lambda_{\beta k}$ ($\beta = 1, 2$), where the subset $\Lambda_{\beta r}$ contains a finite number $2R$ of eigenvalues which are symmetrically arranged on the real axis, Λ_{1k} is an unbounded, symmetric set consisting of the real eigenvalues and Λ_{2k} is a symmetric, unbounded set of complex eigenvalues (we also assign pure imaginary eigenvalues to the category of complex eigenvalues).

If $k_v \in \Lambda_1$ and $k_p \in \Lambda_2$ are simple eigenvalues, then homogeneous elementary solutions of the first kind (the subscript v) and the second kind (the subscript p) that correspond to them are given by the expressions

$$u_{1\nu} = a_{3\nu}(z)\partial_2 m_{2\nu}, \quad u_{2\nu} = -a_{3\nu}(z)\partial_1 m_{2\nu}, \quad u_{z\nu} = 0 \tag{2.4}$$

$$\Delta m_{2\nu} + k_\nu^2 m_{2\nu} = 0, \quad m_{2\nu} = m_{2\nu}(x_1, x_2)$$

$$u_{1p} = a_{1p}(z)\partial_1 m_{1p}, \quad u_{2p} = a_{1p}(z)\partial_2 m_{1p}, \quad u_{zp} = ik_p a_{2p}(z) m_{1p} \tag{2.5}$$

$$\Delta m_{1p} + k_p^2 m_{1p} = 0, \quad m_{1p} = m_{1p}(x_1, x_2)$$

Theorem. If the spectra Λ_1 and Λ_2 consist solely of simple eigenvalues, then any homogeneous solution (a solution which satisfies homogeneous boundary conditions (1.10)) can be represented by a finite or infinite sum of the elementary solutions (2.4) and (2.5). Here, if S is an unbounded domain, the solutions of the Helmholtz equations for real values of k_ν and k_p can be chosen from the condition of the energy radiation principle [2, 7, 8] and, in the case of complex values, from the attenuation condition.

Remark 1. The proof of an analogous theorem on the completeness of systems of homogeneous elementary solutions, presented earlier in [11] for a static problem, can be transferred to the case under consideration with only slight changes.

3. CRITICAL FREQUENCIES AND MODES

We shall say that a frequency ω_c is critical if, among the eigenvalues $k_n \in \Lambda_j(\omega)$, there is a multiple eigenvalue k_c. We shall call the pair (k_c, ω_c) critical and we shall call the elementary solutions corresponding to the critical pair the critical modes.

Consider the case when $k_c = 0$. Substitution of $k = 0$ into problem (2.2) leads to the following problem for determining the set of critical frequencies

$$(c_{44}a_0')' + \rho\omega^2 a_0 = 0, \quad a_0(z^\pm) = 0 \tag{3.1}$$

Substitution of $k = 0$ into (2.3) leads to two problems for determining the set of critical frequencies

$$(c_{44}a_{01}')' + \rho\omega^2 a_{01} = 0, \quad a_{01}'(z^\pm) = 0 \tag{3.2}$$

$$(c_{33}a_{02}')' + \rho\omega^2 a_{02} = 0, \quad a_{02}'(z^\pm) = 0 \tag{3.3}$$

We will first consider problems (3.1) and (3.2) which, although they are identical, are a consequence of the two different problems (2.2) and (2.3). We call the sets of values $\omega = \omega_r$ $(r = 1, 2, \ldots)$, for which these problems have non-trivial solutions, critical frequencies of the first kind. The corresponding eigenfunctions a_0 and a_{01} are denoted by φ_{0r}. In this case, an eigenvector of problem (2.3) has the form $\mathbf{a}_{0r} = \{\varphi_{0r}, 0\}$. Since $k = 0$ is a multiple eigenvalue, the root subspaces of the initial problem (2.3) are not exhausted by the eigenvector. As the investigations in [6, 7], carried out for a homogeneous plate, have shown, different versions of problem (2.3) are possible depending on the parameters. We will now describe them.

For this purpose, we consider problem (2.3) and construct the equations for determining the associated vectors. We have

$$Z(0, \omega_r)\mathbf{a}_{mr} = \mathbf{F}_{mr}, \quad m = 1, 2, \ldots \tag{3.4}$$

$$\mathbf{F}_{1r} = -\partial_k Z(0, \omega_r)\mathbf{a}_{0r}, \quad \mathbf{F}_{mr} = -\partial_k Z(0, \omega_r)\mathbf{a}_{m-1r} - \tfrac{1}{2}\partial_k^2 Z(0, \omega_r)\mathbf{a}_{m-2r}$$

$$Z(k, w)\mathbf{a} \equiv \{(\mathbf{Ca}) + ik[(\mathbf{Ba})' + \mathbf{B}^*\mathbf{a}'] - k^2\mathbf{Aa} + \rho\omega^2\mathbf{a} = 0, \ (\mathbf{Ca} + ik\mathbf{Ba})|_{z=z^\pm}\}$$

Here ∂_k is a derivative with respect to the eigenvalue parameter k.

Since each of problems (3.4) is an "eigenvalue problem", solutions only exist when the conditions

$$d_{mr} = \int_{z^-}^{z^+} F_{mrj}\bar{a}_{0rj}\,dz = 0$$

are satisfied.

When $m = 1$, it follows from Eq. (3.4) that $\mathbf{a}_{1r} = \{\varphi_{0r}, ia_{1r2}\}$, where a_{1r2} are determined by the solutions of the problem

$$(c_{44}a'_{1r2})' + \rho\omega_r^2 a_{1r2} + (c_{13}\varphi_{0r})' + c_{44}\varphi'_{0r} = 0, \qquad (c_{44}a'_{1r2} + c_{13}\varphi_{0r})|_{z=z^\pm} = 0 \tag{3.5}$$

At the same time

$$d_{1r} = \int_{z^-}^{z^\pm} [(c_{13}a'_{1r2})_{\overline{\varphi}_{0r}} - (c_{44}a_{12})\overline{\varphi}'_{0r} + c_{11}\varphi_{0r}\overline{\varphi}_{0r}]dz \tag{3.6}$$

If $d_{1r} \neq 0$, then the Jordan array is exhausted by the two vectors $\mathbf{a}_{0r}$ and $\mathbf{a}_{1r}$ while, if $d_{1r} = 0$, at least a further pair of associated vectors $\mathbf{a}_{2r}$ and $\mathbf{a}_{3r}$ exists.

We now consider the problem of constructing the elementary vectors (modes) corresponding to the critical pairs $(0, \omega_r)$. In this case, the value $k_c = 0$ is a quadruple value with respect to the set of elementary solutions of the first and second kind if $d_{1r} \neq 0$ and has a higher multiplicity (always even) if $d_{1r} = 0$. In order to obtain the system of differential equations describing the distribution of the amplitudes of the critical modes with respect to the variables x_1, x_2, the technique employed earlier in [12, 13] can be used.

Assertion 2. Suppose $d_{1r} \neq 0$. Then, the set of elementary solutions corresponding to the critical pair $(0, \omega_r)$ is determined by the relations

$$u_\alpha = \varphi_{0r}b_\alpha(x_1, x_2) + a_{2r1}\partial_\alpha\theta, \qquad u_3 = a_{1r2}\theta, \qquad \theta = \partial_1 b_1 + \partial_2 b_2 \tag{3.7}$$

$$(\lambda_r + \mu_r)\partial_1\theta + \mu_r\Delta b_1 = 0, \qquad (\lambda_r + \mu_r)\partial_2\theta + \mu_r\Delta b_2 = 0 \tag{3.8}$$

$$\mu_r = \int_{z^-}^{z^+} c_{66}\varphi_{0r}\overline{\varphi}_{0r}dz, \qquad \lambda_r = d_{1r} - 2\mu_r$$

Here, $a = a_{2r1}$ is the solution of the problem

$$(c_{44}a')' + \rho\omega_r^2 a + (c_{44}a_{1r2})' + c_{13}a'_{1r2} + (c_{12} - c_{66}\lambda_r/\mu_r)\varphi_{0r} = 0$$

$$(c_{44}a' + c_{44}a_{1r2})|_{z=z^\pm} = 0$$

Remark 2. Equations (3.8) are identical in form to the equations of the plane theory of elasticity.

Assertion 3. If $d_{1r} = 0$ (the first special case), then at least a further two associated vectors $a_{2r} = \{-a_{2r1}, 0\}$, $a_{r3} = \{0, -ia_{3r2}\}$ exist, where a_{3r2} is the solution of the problem

$$(c_{44}a'_{3k2})' + \rho\omega_r^2 a_{3k2} + (c_{13}a_{2r1})' + c_{44}a'_{2r1} + c_{44}a_{1r2} = 0$$

$$(c_{44}a'_{3r2} + c_{13}a_{2r1})|_{z=z^\pm} = 0 \tag{3.9}$$

The critical modes in this case are determined by the relations

$$u_\alpha = \varphi_{0r}b_\alpha + a_{2r1}\partial_\alpha\theta, \qquad u_3 = a_{1r2}\theta + a_{3r2}C$$

Here C is an arbitrary constant, and the functions b_1 and b_2 satisfy the equations

$$\Delta(\partial_1^2 b_1 + \partial_1\partial_2 b_2) = 0, \qquad \Delta(\partial_1\partial_2 b_1 + \partial_2^2 b_2) = 0 \tag{3.10}$$

Remark 3. Relations (3.7) and (3.8) admit of particular solutions of the form

$$u_\alpha = \varphi_{0r}C_\alpha, \qquad u_3 = 0$$

where C_α are constants. These solutions describe one of the types of resonances of a layer which it is natural to call a longitudinal resonance.

We will now present some fundamental results for the case when problem (3.5) has a non-trivial solution. We shall call the corresponding values of $\omega = \omega_q$ ($q = 0, 1, \ldots$) critical frequencies of the second kind. An eigenvector $\mathbf{a}_{0q} = \{0, a_{0q2}\}$ and an associated vector $\mathbf{a}_{1q} = \{ia_{1q1}, 0\}$ correspond to each critical frequency of the second kind if

$$d_{2q} \neq 0 \tag{3.11}$$

$$d_{2q} = \int\limits_{z^-}^{z^+} \left[a_{1q1}(c_{13}a'_{0q2}) - a'_{1q1}(c_{44}a_{0q2}) - c_{44}a^2_{0q2}\right]dz$$

where a_{1q1} is the solution of the problem

$$(c_{44}a'_{1q1})' + \rho\omega_q^2 a_{1q1} + c_{44}a_{0q} + c_{13}a'_{0q2} = 0$$

$$(c_{44}a'_{1q1} + c_{44}a_{0q2})|^{\pm}_{z=z} = 0$$

Assertion 4. If condition (3.11) is satisfied, the set of critical modes corresponding to the critical pair $(0, \omega_q)$ is described by the relations

$$u_\alpha = a_{1q1}\partial_\alpha m, \quad u_3 = a_{0q2}m, \quad \Delta m = 0 \tag{3.12}$$

Remark 4. Relations (3.12) admit of a solution of the form

$$u_\alpha = 0, \; u_3 = a_{0q2}C$$

where C is an arbitrary constant. This solution is naturally interpreted as the transverse resonances of an unbounded layer.

Assertion 5. If $d_{2q} = 0$ (the second special case), then at least a further two associated vectors $\mathbf{a}_{2q} = \{0, -a_{2q2}\}$, $\mathbf{a}_{3q} = \{-ia_{3q1}, 0\}$ exist, the components of which are determined by the solutions of the problems

$$(c_{33}a'_{2q2})' + \rho\omega_q^2 a_{2q2} + (c_{13}a_{1q1})' + c_{44}a'_{1q1} + c_{44}a_{0q2} = 0$$

$$(c_{33}a'_{2q2} + c_{13}a_{1q1})|_{z=z^\pm} = 0$$

$$(c_{44}a'_{3q1})' + \rho\omega_q^2 a_{3q1} + (c_{44}a_{2q2})' + c_{13}a'_{2q2} + c_{11}a_{3q1} = 0$$

$$(c_{44}a'_{3q1} + c_{44}a_{2q2})|_{z=z^\pm} = 0$$

and the critical modes are determined by the relations

$$u_\alpha = a_{1q1}\partial_\alpha m + a_{3q1}\partial_\alpha \Delta m, \quad u_3 = a_{0q2}m + a_{2q2}\Delta m, \quad \Delta^2 m = 0$$

Assertion 6. If there are critical frequencies of the first and second kind which are identical in magnitude $\omega_c = \omega_r = \omega_q$ in the spectra Λ_1 and Λ_2 (the third special case), then a longitudinal and a transverse critical mode of the form

$$u_\alpha = a_{0r1}b_\alpha, \quad u_3 = 0, \quad \partial_1 b_1 + \partial_2 b_2 = 0, \quad \partial_2 b_1 - \partial_1 b_2 = 0$$

$$u_\alpha = 0, \quad u_3 = a_{0q2}$$

correspond to the pair of eigenvalues $(0, \omega_c)$.

Remark 5. The fact that the Jordan arrays d_{1r}, d_{2q} become infinite is one of the signs of the existence of such a pair of solutions.

The cases of critical frequencies and modes which have been described do not exhaust all the possibilities. In particular, it can be shown that, if $d_{1r} < 0$ ($d_{2q} < 0$), the dispersion curve $\omega = \omega(k)$, emerging from the point $(0, \omega_r)$ ($(0, \omega_q)$) always has a local minimum with a value $k_c \neq 0$ which is multiple.

4. THE SPECTRA OF THE CRITICAL FREQUENCIES OF TRANSVERSELY INHOMOGENEOUS PLATES WITH A PERIODIC STRUCTURE

We put

$$z^- = 0,\ z^+ = H = Nh + \varepsilon,\ 0 \le \varepsilon < h,\ z_s = hs,\ s = 0,\ 1, \ldots N$$

where N is a natural number and consider the problem

$$(ca') + (\rho\omega^2 - ck^2)a = 0 \tag{4.1}$$

$$a'(0) = a'(H) = 0 \tag{4.2}$$

where $c = c(z)$ and $\rho = \rho(z)$ are periodic functions with period h. It is obvious that problems (2.2), (3.2) and (3.3) are special cases of problem (4.1), (4.2).

Resting on Floquet–Lyapunov theory [14], the general solution of Eq. (4.1) can be represented in the form

$$a = A_1 r_1^s y_1(z - z_s) + A_2 r_2^s y_2(z - z_s),\ z_s \le z \le z_{s+1} \tag{4.3}$$

where A_1 and A_2 are arbitrary constants, $z_s = sh(s = 0, 1, \ldots, N-1)$, $y_1(z)$, $y_2(z)$ are a pair of linearly independent solutions of Eq. (4.1) in the interval $z \in [0, h]$ and r_1, $r_2 = r_1^{-1}$ are the roots (multipliers) of the recurrent equation

$$r^2 - 2b(k,\omega)r + 1 = 0;\ r_1 = b + (b^2 - 1)^{1/2} \tag{4.4}$$

Remark 6. If we put $k = 0$, $r = e^{i\beta h}$ in Eq. (4.4), we obtain the dispersion equation $\cos\beta = b(\omega)$ of the normal modes in an unbounded stratified medium which, according to Floquet's theory, can be represented in the form $u = e^{i\beta z}y(z)$, where $y(z)$ is an h-periodic function. At the same time, $b(\omega) < 1$ determines the transmission bands and the condition $b(\omega) > 1$ determines the cut-off bands [15]. The case when $c = c_{33}$ corresponds to a longitudinally polarized waves and the case when $c = c_{44}$ corresponds to transversely polarized waves.

Substituting expression (4.3) into boundary conditions (4.2), we obtain

$$y_1'(0)A_1 + y_2'(0)A_2 = 0,\ r_1^N y_1'(\varepsilon)A_1 + r_1^{-N} y_2'(\varepsilon)A_2 = 0 \tag{4.5}$$

It follows from the condition for a non-trivial solution of system (4.5) to exist that

$$y_2'(0)y_1'(\varepsilon)r_1^N - y_1'(0)y_2'(\varepsilon)r_1^{-N} = 0 \tag{4.6}$$

In the problem under investigation, the case when $b(k, \omega) < 1$ is of interest. In this case, $A_2 = \bar{A}_1$, $y_2 = \bar{y}_1$. At the same time, as a result of the substitution $r_1 = e^{i\beta h}$, Eq. (4.6) is transformed into the following equation

$$\sin(N\beta h + \tau) = 0,\ \operatorname{tg}\tau = \operatorname{Im} w/\operatorname{Re} w,\ w = y_1'(\varepsilon)\bar{y}_1'(0) \tag{4.7}$$

We denote by $\beta_m = (m\pi - \tau)/hN$ $(m = 0, 1, \ldots)$ the roots of Eq. (4.7). Returning to Eq. (4.4) and taking account of the fact that

$$r + r^{-1} = 2\cos(h\beta),\ \cos(h\beta_{2N+m}) = \cos(h\beta_m)$$

we obtain $2N$ dispersion equations

$$F_j(k,\omega) = b(k,\omega) - \cos(h\beta_j) = 0,\ j = 0, 1, \ldots, 2N - 1 \tag{4.8}$$

In the case when $k = 0$, $c = c_{44}$ $(F_j(0, \omega) = F_{1j}(0, \omega))$, relations (4.8) determine the spectrum S_1 of critical frequencies of the first kind and, in the case when $k = 0$, $c = c_{33}$ $(F_j(0, \omega) = F_{2j}(0, \omega))$, they determine the spectrum S_2 of critical frequencies of the second kind. It also follows from relations (4.8) that, in the case of arbitrary piecewise-continuous functions $c(z)$ and $\rho(z)$, we have

$$S_\alpha = \bigcup_{j=0}^{2N-1} S_{\alpha j}, \quad \alpha = 1, 2$$

where $S_{\alpha j}$ is the set of roots of the functions $F_{\alpha j}(0, \omega)$. If $H = Nh$, then $\tau = 0$. In this case

$$\cos(h\beta_{2N-n}) = \cos(h\beta_n), \quad n = 0, 1, \ldots, N$$

5. SOME RESULTS OF A NUMERICAL ANALYSIS FOR A FINELY STRATIFIED PLATE

Consider the case when

$$c = c^{(1)}, \rho = \rho^{(1)} \text{ when } z_s \leqslant z \leqslant z_s + h_1$$

$$c = c^{(2)}, \rho = \rho^{(2)} \text{ when } z_s + h_1 \leqslant z \leqslant z_{s+1}$$

Here $c^{(1)}$, $\rho^{(1)}$, $c^{(2)}$ and $\rho^{(2)}$ are constants. We shall call a structure which satisfies these conditions a finely stratified structure [16, 17]. In this case

$$b(k, \omega) = \cos\gamma_1 h_1 \cos\gamma_2 h_2 - \tfrac{1}{2}(p + p^{-1})\sin\gamma_1 h_1 \sin\gamma_2 h_2$$

$$\gamma_1 = \sqrt{k_1^2 - k^2}, \quad \gamma_2 = \sqrt{k_2^2 - k^2}, \quad k_\alpha = \omega / v^{(\alpha)}, \quad v^{(\alpha)} = (c^{(\alpha)} / \rho^{(\alpha)})^{1/2}$$

$$p = c_2\gamma_2/(c_1\gamma_1), \quad h_2 = h - h_1$$

The expressions for the eigenfunctions and associated functions are not presented here in view of their length.

The calculations were carried out for a steel – rubber pair:

For steel: $\rho^{(1)} = 7.8 \times 10^{-6}$ kg/m^3; $v_t^{(1)} = 3.17 \times 10^5$ m/s; $v_l^{(1)} = 5.83 \times 10^5$ m/s;

For rubber: $\rho^{(2)} = 1.2 \times 10^6$ kg/m^3; $v_t^{(2)} = 1.28 \times 10^5$ m/s; $v_l^{(2)} = 9.13 \times 10^5$ m/s.

We will now explain the notation which is used below: $\Omega = \omega h / v_l^{(1)}$ is the reduced frequency; Ω_{1nm}, Ω_{2nm} are the values of the critical reduced frequencies belonging to S_{1n} and S_{2n}; a_{1nm} and a_{2nm} are the corresponding natural modes; λ_{im} and μ_{im} are the pseudoelastic constants which appear in Eqs (3.8); $\xi = h_1/h$ is the dimensionless thickness of the steel layer; ξ_{1kr} and ξ_{2pq} are the values of the parameter ξ for which the invariants $d_{1kr} = 0$, $d_{2pq} = 0$ (see Assertions 3 and 5); η_{1kr} and η_{2pq} are the values of the parameter ξ at which d_{1kr} or d_{2pq} respectively become infinite (see Remark 5).

The values of Ω_{1nm} and Ω_{2nm}, calculated for $\xi = 0.5$ when $N = 8$, $n = 0, 1, 2, \ldots, 8$ are shown in Table 1. They are identical to the values when $N = 16$, which correspond to $n = 0, 2, 4, \ldots, 16$. The rows illustrate the frequency distribution within the transmission band. Here, the lowest frequency is the lower limit of the band. The values of the reduced frequencies, calculated using the formulae

$$\Omega_{1n} = K_1 \frac{n\pi}{N}, \quad \Omega_{2n} = K_2 \frac{n\pi}{N}, \quad K_1 = v_t^{(1)} \frac{\langle \rho \rangle}{\langle c_{44} \rangle}, \quad K_2 = v_l^{(1)} \frac{\langle \rho \rangle}{\langle c_{33} \rangle}$$

$$\langle \rho \rangle = \frac{1}{h}\int_0^h \rho(z)dz, \quad \langle c_{ss} \rangle = \frac{1}{h}\int_0^h \frac{dz}{c_{ss}}$$

which follow from the theory of averages, are separated out into a single row. They give a certain idea on the domain of applicability of this theory for determining the characteristic frequencies of elastic elements made from strongly inhomogeneous media.

The values of the pseudoelastic constants λ_{1n} and μ_{1n} for the critical frequencies from the first row of Table 1 are presented in Table 2. Special cases were analysed with respect to the parameter ξ for the frequencies Ω_{111} and Ω_{211} when $N = 8$ and showed that one value each of $\xi_{111} = 0.9232$, $\eta_{111} = 1.171$ exists but two values each of $\xi_{211} = 0.06748, 0.3022$, $\eta_{211} = 0.08868, 0.6704$.

As an illustration of the amplitude distribution of the oscillations throughout the thickness, graphs of the natural modes a_{211} (the solid curve), a_{221} (the dashed curve) and a_{212} (the dot-and-dash curve) as a function of z are shown in Fig. 1.

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Table 1

n	0	1	2	3	4	5	6	7	8
Ω_{1n1}	0	0.1137	0.2242	0.3281	0.4219	0.5015	0.5625	0.6012	0.6144
Ω_{1n2}	2.660	2.653	2.632	2.600	2.563	2.525	2.492	2.470	2.462
Ω_{1n3}	4.809	4.819	4.846	4.889	4.941	4.997	5.047	5.083	5.096
Ω_{1n}	0	0.1142	0.2284	0.3426	0.4568	0.5709	0.6851	0.7993	0.9135
Ω_{2n1}	0	0.3818	0.7581	1.123	1.466	1.776	2.031	2.204	2.266
Ω_{2n2}	6.790	6.714	6.516	6.253	5.968	5.695	5.461	5.298	5.240
Ω_{2n3}	8.768	8.849	9.059	9.345	9.665	9.991	10.29	10.53	10.63
Ω_{2n}	0	0.3826	0.7652	1.148	1.531	1.913	2.296	2.678	3.061

Table 2

n	1	2	3	4	5	6	7	8
Ω_{1n1}	0.1137	0.2242	0.3281	0.4219	0.5015	0.5625	0.6012	0.6144
$\lambda_{1n} \times 10^{-3}$	4998	2877	2853	3661	3795	3505	3299	3244
$\mu_{1n} \times 10^{-3}$	2499	1439	1427	1830	1898	1753	1649	1622

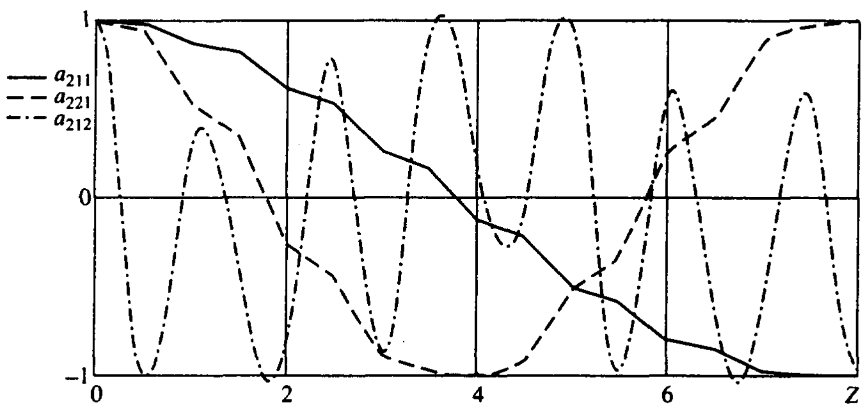

Fig. 1

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